A Dynamic Programing Approach to Optimal Stochastic Orbital Transfer Strategy

T. Nishimura*
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif.

AND

C. G. Pfeiffer†
TRW Systems, Redondo Beach, Calif.

The optimal stochastic orbit transfer strategy is defined as the sequence of guidance corrections which will minimize a statistical measure of final error, subject to the constraint that the total correction capability expended be less than a specified number. The dynamic programing algorithm is employed to solve this problem. The numerical difficulty of storing the many values of the optimized performance index corresponding to every discrete value of the state variables is overcome by representing the performance surface only in the neighborhood of local minima. The computer program designed to solve this problem is described, and some numerical results applicable to a space mission of a Mars orbiter are presented.

Introduction

An important objective of some space missions is to achieve a specified elliptical orbit around a target body (e.g., Apollo, Martian, and Lunar Orbiters). After the insertion maneuver, however, the spacecraft may reach a dispersed orbit due to orbit determination and maneuver execution errors, and a sequence of guidance corrections (possibly only two) becomes mandatory in order to accomplish the mission. The guidance strategy is the specification of how many corrections should be applied, when, and what they should be.

The determination of a strategy that minimizes a statistical measure of the final orbit error in the presence of orbit determination (estimation) and guidance execution errors poses an important unsolved problem. The classical optimal orbit transfer analysis, which seeks the strategy which minimizes propellant (correction capability) expenditure and does not consider random errors,[1] can at best yield an approximately optimal strategy. Instead, we will assume that there is a constraint that the total correction capability expended during the mission be less than a specified number (the resource initially allotted), and we will seek the strategy which minimizes the expected value of a weighted sum of squares of the final orbit errors. These dispersions arise from random estimation and correction execution errors.

Several authors[2-4] have dealt with a problem of this type and obtained some interesting results by analysis of simplified cases or by developing a suboptimal guidance strategy. This paper will employ the dynamic programing algorithm[5] to formulate a technique for determining the optimal stochastic orbital transfer strategy. The well-known numerical difficulty of storing the many values of the optimized performance index corresponding to every discrete value of the state variables is overcome by recognizing that the only regions of the performance surface that are of real interest are the neighborhoods of the local minima which occur. Thus, for each guidance correction we need to store only the coordinates of the local minima, the associated optimal performance index and expected value of the correction capability required to complete the mission, and a quadratic approximation of the local behavior of the performance surface.

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* Member of Technical Staff, Systems Analysis Research Section.

† Head of Mathematical Physics Section, Guidance Analysis Department.

Determination of the Optimal Final Correction

Let the statistical measure of mission success at completion of the final (Nth) guidance correction be

$$J_1(\mathbf{x}_N,\Delta\mathbf{u}_N,\theta_N) = E[(\mathbf{x}_N + \Delta\mathbf{x}_N(\Delta\mathbf{u}_N,\theta_N) - \mathbf{x}_d)^T W(\mathbf{x}_N + \Delta\mathbf{x}_N(\Delta\mathbf{u}_N,\theta_N) - \mathbf{x}_d)] \quad (1)$$

where $\mathbf{x}_N$ = the state vector just prior to the Nth correction, consisting of orbit parameters; $\Delta\mathbf{x}_N$ = the change resulting from the correction; W = an a priori specified semipositive-define weighting matrix; $\mathbf{x}_d$ = the state of the desired orbit; $\mathbf{u}_{cN}$ = the correction applied; $\Delta\mathbf{u}_N$ = the resulting velocity change; θ_N = the true anomaly defining the position on the orbit where the correction is applied. $E\ldots$ = a statistical expectation.

The θ_N and the components of $\Delta\mathbf{u}_N$ are the control variables to be determined. The geometry of the orbit is such that, in general, $\mathbf{x}_N$ cannot be made equal to $\mathbf{x}_d$ by performing a single correction $\Delta\mathbf{u}_N$. Applying assumptions usually made in orbit determination and guidance analysis, we suppose that a maximum likelihood estimate of $\mathbf{x}_N$ (estimate denoted by star) is available and, for a given value of θ_N, that the error in the estimate is approximately gaussian,‡ that is

$$\boldsymbol{\varepsilon}_N(\theta_N) = [\mathbf{x}_N^*(\theta_N) - \mathbf{x}_N(\theta_N)] = \{\text{gaussian, mean zero, covariance known}\}$$

Furthermore, we suppose that for any given correction, the error in executing the correction results in a state error which

‡ This assumption can be justified if the estimate is based upon many data points, for the maximum likelihood estimate error distribution is asymptotically gaussian.

is approximately gaussian, that is

$$\mathbf{n}_N(\text{given } \Delta\mathbf{u}_N,\ \theta_N,\ \mathbf{x}_N^*) = [\Delta\mathbf{x}_N^* - \Delta\mathbf{x}_N] = \{\text{guassian, mean zero, covariance known}\}$$

where $\Delta\mathbf{x}_N^*$ denotes the commanded (expected) value of $\Delta\mathbf{x}_N$. The $\mathbf{n}_N$ depends upon θ_N and the magnitude and direction of $\Delta\mathbf{u}_N$, and is zero if no correction is made. Then, assuming $\mathbf{n}_N$ and $\boldsymbol{\varepsilon}_N$ to be independent, we have

$$J_1 = (\mathbf{x}_N^* + \Delta\mathbf{x}_N^* - \mathbf{x}_d)^T W(\mathbf{x}_N^* + \Delta\mathbf{x}_N^* - \mathbf{x}_d) + E[\boldsymbol{\varepsilon}_N^T W \boldsymbol{\varepsilon}_N] + E[\mathbf{n}_N^T W \mathbf{n}_N] \quad (2)$$

If the correction capability available is c_N, the optimal final correction is then found from solving

$$p_1(\mathbf{x}_N^*,c_N) = \begin{cases} \underset{(\Delta\mathbf{u}_N,\theta_N)}{\text{minimum}}\ [J_1(\mathbf{x}_N^*,\Delta\mathbf{u}_N,\theta_N)] & \text{if } |\Delta\mathbf{u}_N| < c_N \\ f_1(\mathbf{x}_N^*,c_N) & \text{if } |\Delta\mathbf{u}_N| \geq c_N \end{cases} \quad (3)$$

where $p_1(\mathbf{x}_N^*,c_N)$ is the single stage optimized performance index. In general $p_j(\mathbf{x}_{N-j+1},{}^*c_{N-j+1})$ is a j-stage optimized performance index given an orbit $\mathbf{x}_{N-j+1}$. Also $f_1(\mathbf{x}_N^*,c_N)$ is a specified penalty function reflecting the result of a single-correction policy to be applied when correction capability is not adequate to complete the mission. Normally a large number is assigned to it as a penalty, so that maneuvers exceeding the propellant budget can be eliminated. This representation assumes that two cases apply: correction capability limited, where a heuristic performance index is adequate, and correction capability unlimited, where the optimization algorithm must be applied. (This is a reasonable approach, because for most missions the capability limited case rarely occurs.) In the latter case the $\Delta\mathbf{u}_N$ and θ_N which minimize J_1 must be obtained numerically for each $\mathbf{x}_N^*$, for the final state is a nonlinear function of $\mathbf{x}_N$, $\Delta\mathbf{u}_N$, and θ_N. Note that the $\mathbf{x}^*$ and c (rather than $\mathbf{x}$ and c) are the state variables for this problem.

Determination of the Correction Sequence

The procedure for determining $p_1(\mathbf{x}_N^*,c_N)$ can be applied for a large number of values of c_N and sample vectors $\mathbf{x}_N^*$, a multidimensional table of the results can be stored, and, applying the dynamic programing algorithm, the corrections $\Delta\mathbf{u}_{N-1}$, θ_{N-1} and the optimized performance index $p_2(\mathbf{x}_{N-1}^*, c_{N-1})$ can be determined. This is a formidable task, for the number of values to be stored is (number of vectors $\mathbf{x}_N^*$) $\times$ (number of components of $\mathbf{x}_N^*$) $\times$ (number of values of c_N). The storage requirements can be greatly reduced if the random errors are small, however, for then we are only interested in values of the performance index and the associated expected correction capability required to complete the mission in the neighborhoods of the local minima of the performance surface. To simplify notation let

$$p_1(\mathbf{x}_N^*,c_N = \infty) = p_1(\mathbf{x}_N^*) \quad (4a)$$

$$EC_N = \text{expected value of correction capability to optimally complete the mission, given } \mathbf{x}_N^* \quad (4b)$$

Since we are supposing that $f_1(\mathbf{x}_N^*,c_N)$ can be a priori specified, we seek to determine and represent $p_1(\mathbf{x}_N^*)$ and EC_N in the regions of local minima.

The $p_1(\mathbf{x}_N^*)$ surface must attain its absolute minimum at $\mathbf{x}_N^* = \mathbf{x}_d$, for W is semipositive definite. The neighborhood of this point is a quadratic region for the Nth correction. On the other hand, there must be other local minima if $\mathbf{x}_N$ is a nonlinear function of $\Delta\mathbf{u}_N$, and these minima must be considered when determining the correction sequence. Considering the effects of execution errors and correction capability constraints, and the fact that it may not be physically possible to attain the desired orbit with a single correction, it may be desirable to aim for an intermediate orbit (i.e., an aiming point not equal to $\mathbf{x}_d$) specified by the coordinates of a local minimum rather than $\mathbf{x}_d$. Thus, we shall represent the function $p_1(\mathbf{x}_N^*)$ the following way: let $\tilde{x}_{1i}$ be the coordinates of the m local minima ($i = 1 \ldots m$); let β_{1i} be the corresponding m values of the optimized performance index; let $\tilde{W}_{Ni}$ be the m weighting matrices describing curvature of the $p_1(\mathbf{x}_N^*)$ surface in the neighborhood of the $\tilde{x}_{1i}$, determined from a local quadratic fit to the p_1 surface and let Ec_{Ni} be the expected correction capability required to complete the mission optimally if $\mathbf{x}_N^* = \tilde{x}_{Ni}$. Then in a small neighborhood of $\tilde{x}_{Ni}$ we have

$$p_{1i}(\mathbf{x}_N^*) \cong \beta_{1i} + (\mathbf{x}_N^* - \tilde{x}_{Ni})^T \tilde{W}_{Ni}(\mathbf{x}_N^* - \tilde{x}_{Ni}) \times \begin{Bmatrix} \mathbf{x}_N^* \text{ in region } i \\ c_N > Ec_{Ni} \end{Bmatrix} \quad (5)$$

This approximation represents $p_1(\mathbf{x}_N^*)$ in the important regions of $\mathbf{x}_N^*$ space, and requires that fewer numbers be stored. For example, if the dimension of $\mathbf{x}_N^*$ is 3 we have (since $\tilde{W}_{Ni}$ is a symmetric matrix)

number of stored values to represent $p_1(\mathbf{x}_N^*)$

$$= [(6 \text{ numbers for } \tilde{W}_{Ni}) + (3 \text{ numbers for } \tilde{x}_{Ni}) + (1 \text{ number for } \beta_{1i}) + (1 \text{ number for } Ec_{Ni})] \times [m] = 11 \cdot m \text{ numbers}$$

The algorithm for finding $p_2(\mathbf{x}_{N-1}^*)$ can now be much simplified, for applying the results of the proceeding part, we replace J_2 with $p_1(\mathbf{x}_N^*)$ as given by Eq. (5) to obtain

$$p_2(\mathbf{x}_{N-1}^*,c_{N-1}) = \begin{cases} \begin{Bmatrix}\text{minimum} \\ \Delta\mathbf{u}_{N-1},\theta_{N-1},i\end{Bmatrix} [p_{1i}(\mathbf{x}_{N-1}^*,\Delta\mathbf{u}_{N-1},\theta_{N-1})] & \text{if } |\mathbf{u}_{c,N-1}| + Ec_{Ni} < c_{N-1} \\ f_2(\mathbf{x}_{N-1}^*,c_{N-1}) & \text{if } |\mathbf{u}_{c,N-1}| + Ec_{Ni} \geq c_{N-1} \end{cases} \quad (6)$$

This procedure can be applied to any number of earlier corrections, and, hence, the complete correction sequence can be determined as a function of x_{N-k}^*. The significant results obtained by this analysis are the coordinates of the local minima to be aimed for as a function of $\mathbf{x}_{N-k}^*$, for these specify the intermediate orbits and the number of corrections to apply. For example, essentially a single correction policy would result if one always sought the minimum centered at $\mathbf{x}_d$; otherwise multiple corrections would be necessary. In order to obtain the numerical results presented in the example, the coordinates of the local minima $\mathbf{x}_{Ni}^*$ were found by generating and plotting a large number of values of $p_1(\mathbf{x}_N^*)$, and the $\tilde{W}_{Ni}$ were determined from a quadratic fit to the p_1 surface in the neighborhood of these points.

State and Control Variables for the Planar Problem

We confine ourselves in this paper to the problem of orbital transfer between two coplanar elliptic orbits. Coordinates of a spacecraft along an elliptic orbit around the planet are described by the following four parameters: θ, the true anomaly; T, the orbital period; h_p, the periapsis altitude; and ω, the angular rotation of line of apsides. The last three parameters form a state vector $\mathbf{x}$ to be controlled by this program. Let the desired target orbit be described by a vector $\mathbf{x}_d$, where

$$\mathbf{x}_d = \begin{bmatrix} T_d \\ h_{pd} \\ \omega_d \end{bmatrix} \quad (7)$$

The velocity vector $\mathbf{u}$ of the spacecraft is defined by a speed component v and a path angle γ (Fig. 1) (γ is measured counterclockwise).

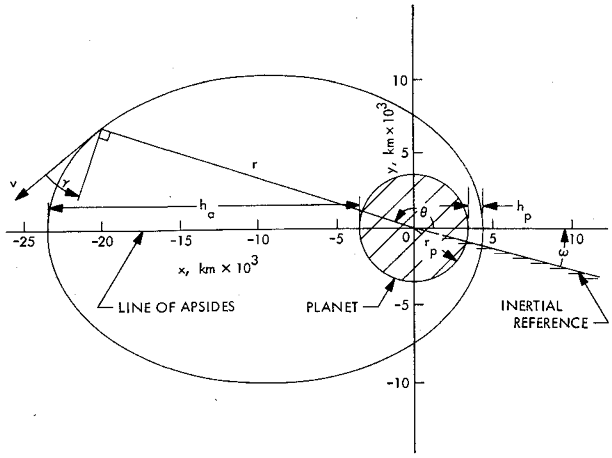

Fig. 1 Definition of orbital parameters.

Suppose the initial velocity $\mathbf{u}_0$ is modified to $\mathbf{u}_1$ as a result of an impulsive velocity correction $\mathbf{u}_c$ at an initial true anomaly θ_0 where

$$\mathbf{u}_1 = \begin{bmatrix} v_1 \\ \gamma_1 \end{bmatrix} = \begin{bmatrix} v_0 + \Delta v \\ \gamma_0 + \Delta\gamma \end{bmatrix} \tag{8}$$

The quantities Δv, $\Delta\gamma$, and θ_0 are the control variables. The components of $\mathbf{u}_c$ are obtained as a vector difference between $\mathbf{u}_1$ and $\mathbf{u}_0$ (Fig. 2)

$$\mathbf{u}_c = \begin{bmatrix} v_c \\ \alpha \end{bmatrix} = \mathbf{u}_1 - \mathbf{u}_0 \tag{9}$$

where

$$v_c = (v_1^2 + v_0^2 - 2v_0v_1 \cos\Delta\gamma)^{1/2} \tag{10}$$

$$\alpha = \sin^{-1}\left(\frac{v_1 \sin\Delta\gamma}{v_c}\right) \qquad -\pi \leq \alpha \leq \pi \tag{11}$$

The state vector $\mathbf{x}_0$ is transferred to $\mathbf{x}_1$ as a result of this correction $\mathbf{u}_c$

$$\mathbf{x}_1 = \begin{bmatrix} T_1 \\ h_{p1} \\ \omega_1 \end{bmatrix} \tag{12}$$

The components of $\mathbf{x}_1$ are computed from $\mathbf{x}_0$ and $\mathbf{u}_0$ through the following relations

$$T_1 = 2\pi(a_1^3/\mu)^{1/2} \tag{13}$$

$$h_{p1} = a_1(1 - e_1) - r_p \tag{14}$$

$$\omega_1 = \omega_0 + \theta_0 - \theta_1 \tag{15}$$

where

$$a_1 = r\mu/(2\mu - rv_1^2) \tag{16}$$

$$e_1 = [1 - (r^2v_1^2/a_1\mu)\cos^2\gamma_1]^{1/2} \tag{17}$$

$$\theta_1 = \cos^{-1}\{[a_1(1 - e_1^2) - r]/re_1\} \tag{18}$$

and symbols used are defined as follows: r = radial distance from the center of the planet; μ = gravitational constant of the planet; r_p = average radius of the planet; a_1 = magnitude of semimajor axis after correction; e_1 = eccentricity of the orbit after correction; and θ_1 = true anomaly of the vehicle after correction.

Stochastic Error Model

Two kinds of stochastic errors may be considered, namely an estimation error of orbital parameters and an execution error of correction. The estimation error is neglected for the example discussed here by assuming that it may be reduced to a sufficiently small value should the spacecraft be permitted to orbit the planet for an adequate period before the correction. It is not difficult to modify the analysis when this assumption is removed.

As for the execution error, only a scale factor error and a pointing error are taken into account. The scale factor error is in the direction of $\mathbf{u}_c$ and is proportional to $\mathbf{u}_c = v_c$. The pointing error is in the direction perpendicular to $\mathbf{u}_c$ and is proportional to v_c. Let s and p be dimensionless random variables associated with these two execution errors having zero mean and variances σ_s^2 and σ_p^2, respectively. Let $\mathbf{e}$ be an execution error vector consisting of e_v and e_γ, respectively, the error in the final values of v and γ.

$$\mathbf{e} = \begin{bmatrix} e_v \\ e_\gamma \end{bmatrix} \tag{19}$$

Then these two components are described by the following equations:

$$e_v = sv_c \cos(\alpha - \Delta\gamma) - pv_c \sin(\alpha - \Delta\gamma) \tag{20}$$

$$e_\gamma = s(v_c/v_1)\sin(\alpha - \Delta\gamma) + p(v_c/v_1)\cos(\alpha - \Delta\gamma) \tag{21}$$

Using the expressions for v_c and α of Eqs. (10) and (11), the equations for $\mathbf{e}$ can be simplified and be expressed in the following vector form

$$\mathbf{e} = B\mathbf{s} \tag{22}$$

where B is a 2×2 matrix with

$$B_{11} = (1 - \cos\Delta\gamma)v + \Delta v \tag{23}$$

$$B_{12} = -v \sin\Delta\gamma \tag{24}$$

$$B_{21} = (1 - \Delta v/v)\sin\Delta\gamma \tag{25}$$

$$B_{22} = (1 - \cos\Delta\gamma + \Delta v/v)(1 - \Delta v/v) \tag{26}$$

and $\mathbf{s}$ is a stochastic variable vector composed of

$$\mathbf{s} = \begin{bmatrix} s \\ p \end{bmatrix} \tag{27}$$

Iterative Method for Finding the Single-Stage Optimal Control

An iterative technique (Newton's method) is employed in order to find an optimal control $\Delta\mathbf{u}$ composed of

$$\Delta\mathbf{u} = \begin{bmatrix} \Delta v \\ \Delta\gamma \end{bmatrix} \tag{28}$$

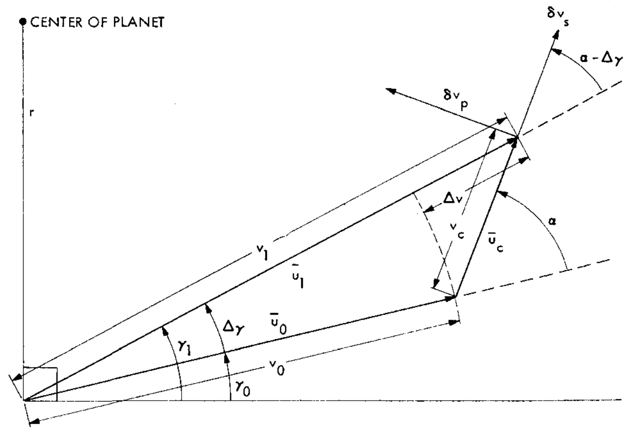

Fig. 2 Initial and final velocity vectors and impulsive correction vector.

The final state $\mathbf{x}_1$ is expanded into Taylor series with respect to $\Delta\mathbf{u}$ and the higher order terms are dropped. Then

$$\mathbf{x}_1 \approx \mathbf{x}_0 + \Delta\mathbf{x} + \mathbf{n} \quad (29)$$

where

$$\Delta\mathbf{x} \cong (\partial\mathbf{x}/\partial\mathbf{u})\Delta\mathbf{u} = X_u\Delta\mathbf{u} \quad (30)$$

$$\mathbf{n} = (\partial\mathbf{x}/\partial\mathbf{u})\mathbf{e} = X_u\mathbf{e} \quad (31)$$

where $\mathbf{n}$ is a random disturbance to the state vector caused by execution errors and X_u is a (3×2) matrix of partials

$$X_u = \partial\mathbf{x}/\partial\mathbf{u} = \begin{bmatrix} \partial T/\partial v, & \partial T/\partial\gamma \\ \partial h_p/\partial v, & \partial h_p/\partial\gamma \\ \partial\omega/\partial v, & \partial\omega/\partial\gamma \end{bmatrix} \quad (32)$$

These components of the X_u matrix are obtained by differentiating the elements of vector $\mathbf{x}_1$ specified by Eqs. (13–15) together with the auxiliary variables a_1, e_1 and θ_1 of Eqs. (16–18) with respect to v and γ. Thus

$$\partial T/\partial v = 6\pi\mu v(a/\mu)^{5/2}, \; \partial T/\partial\gamma = 0 \quad (33)$$

$$\partial h_p/\partial v = [2a^2(1 - e)^2/vr\cdot(1 - \cos\theta)]/(1 + e\cos\theta) \quad (34)$$

$$\partial h_p/\partial\gamma = -[a(1 - e^2)/e]\tan\gamma \quad (35)$$

$$\partial\omega/\partial v = 2\sin\theta/ev \quad (36)$$

$$\frac{\partial\omega}{\partial\gamma} = \frac{2e + (1 + e^2)\cos\theta}{e(1 + e\cos\theta)} \quad (37)$$

Substituting the expression of $\mathbf{x}_1$ in Eq. (29) into Eq. (2), the performance index becomes

$$J_1(\mathbf{x}_0,\Delta\mathbf{u}_0,\theta_0) = (\mathbf{x}_0 + \Delta\mathbf{x}_0 - \mathbf{x}_d)^T W(\mathbf{x}_0 + \Delta\mathbf{x}_0 - \mathbf{x}_d) + f_0 \quad (38)$$

where f_0 is the contribution of the execution errors to the performance index

$$f_0 = E(\mathbf{n}^T{}_0 W\mathbf{n}_0) = E(\mathbf{e}^T X_u{}^T W X_u\mathbf{e}) = E(\mathbf{s}^T B^T X_u{}^T W X_u B\mathbf{s}) \quad (39)$$

and is called the stochastic dispersion factor.

Let

$$W_j = B^T X_u{}^T W X_u B \quad (2 \times 2) \quad (40)$$

and assume that the random variables s and p are independent of each other

$$E[s\cdot p] = 0 \quad (41)$$

Then f_0 is reduced to

$$f_0 = \sigma_s{}^2 W_{j11} + \sigma_p{}^2 W_{j22} \quad (42)$$

The derivative of J with respect to $\mathbf{u}$ is set equal to zero in order to minimize the performance index J. Using the first-order approximation of $\mathbf{x}_1$,

$$0 \simeq (\partial x/\partial u)^T\, W[\mathbf{x}_0 + (\partial\mathbf{x}/\partial\mathbf{u})\delta\mathbf{u}_0 - \mathbf{x}_d] + \mathbf{g}_0 \quad (43)$$

where $\mathbf{g}_0$ is the column portion of $\partial f_0/\partial\mathbf{u}$.

Then the approximate solution of Eq. (43) becomes

$$\delta\mathbf{u}_0 = -(X_u{}^T W X_u)^{-1}[X_u{}^T W(\mathbf{x}_0 - \mathbf{x}_d) + \mathbf{g}_0] \quad (44)$$

The true anomaly is initially fixed at θ_{0i}, $i = 1, \ldots, N$ along the initial orbit, and $\delta\mathbf{u}_0$ is calculated at each point from Eq. (44). Let these values be denoted $\delta\mathbf{u}_0(1)$. Then for each θ_{0i}, a new velocity vector $\mathbf{u}_0(1)$ is computed by

$$\mathbf{u}_0(1) = \mathbf{u}_0 + \delta\mathbf{u}_0(1) \quad (45)$$

and a new state vector $\mathbf{x}_0(1)$ and the true anomaly $\theta_{0i}(1)$ are computed by means of Eqs. (13–18). The $\delta\mathbf{u}_0(1)$ is not the optimal control to be carried out, but it is only an initial guess. Having the updated parameters $\mathbf{x}_0(1)$, $\mathbf{u}_0(1)$ and $\theta_{0i}(1)$, the next fraction of control $\delta\mathbf{u}_0(2)$ is computed by Eq. (45), and is added to $\mathbf{u}_0(1)$ to generate $\mathbf{u}_0(2)$. Thus, the process is iterated until $\mathbf{u}_0(i)$ converges to some value $\mathbf{u}_1{}^*$.

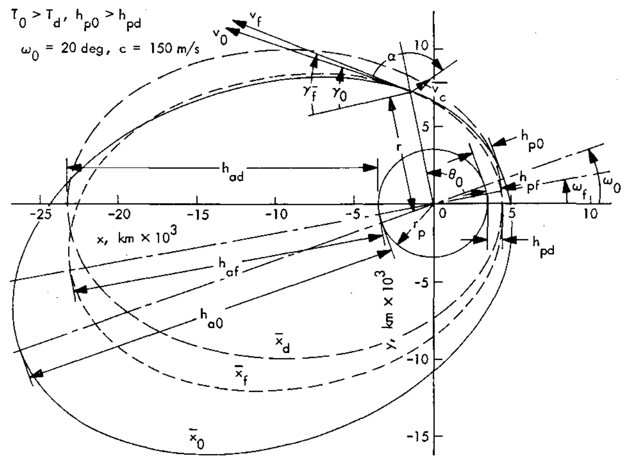

Fig. 3 Optimal orbit transfer.

Hence, for each θ_{0i},

$$\mathbf{u}_1{}^* = \mathbf{u}_0 + \Delta\mathbf{u}_0{}^*, \; \Delta\mathbf{u}_0{}^* = \sum_{j=1}^{m} \delta\mathbf{u}_0(\mathrm{j}) \quad (46)$$

This $\Delta\mathbf{u}_0{}^*$ is the optimal control to be applied at the given true anomaly θ_{0i}. The resulting $\mathbf{x}_1$ and true anomaly θ_{1i} are computed by Eqs. (13–18) using the above $\mathbf{u}_1{}^*$.

General Description of Single-State and Two-Stage Programs

This procedure provides the optimal control at specified values of initial true anomaly (θ_0) along the orbit. In order to acquire the over-all optimal maneuver, a search for the minimum performance index as a function of θ_0 is carried out. This subroutine essentially constructs two tables, one for a rough mesh of θ_0 and the other for a fine mesh of θ_0. The first table is a collection of $p_1(\mathbf{x}_0,\theta_0)$ supplied by the minimization subroutine at the rough mesh points $\theta_0 = i\cdot\Delta\theta$, $i = 1, 2, \ldots, N$, e.g., $\Delta\theta = 10°$. (See Fig. 3.)

The program searches for the minimum p_1 in the first table and the associated true anomaly θ_x. Then the second table of a fine mesh $\Delta\Delta\theta$ (e.g., 0.5°) is obtained around this θ_x within the range of $\pm\Delta\theta$ and the overall minimum point is derived from that table. This process is mathematically described, including the optimization by $\Delta\mathbf{u}_0$ of the previous part, as

$$p_1(\mathbf{x}_0) = \min_{\theta_0,\Delta\mathbf{u}_0} J_1(\mathbf{x}_0,\Delta\mathbf{u}_0,\theta_0) \quad (47)$$

This program of finding the optimal control through single maneuver which involves two minimization steps (first by $\Delta\mathbf{u}_0$, then by θ_0) described by Eq. (47) is called the "Optimal Stochastic Transfer–Single-Stage (OSTSS) Program" (see Fig. 4a).

When two maneuvers $\Delta\mathbf{u}_0$ and $\Delta\mathbf{u}_1$ are permitted, the control strategy is again to minimize the mean squared distance between the final state $\mathbf{x}_2$ and the target state $\mathbf{x}_d$

$$p_2(\mathbf{x}_0) = \min_{\theta_0,\Delta\mathbf{u}_0,\theta_1,\Delta\mathbf{u}_1} E[(\mathbf{x}_2 - \mathbf{x}_d)^T W(\mathbf{x}_2 - \mathbf{x}_d)]$$
$$= \min_{\theta_1,\Delta\mathbf{u}_0,\theta_1,\Delta\mathbf{u}_1} J_1(\mathbf{x}_1,\Delta\mathbf{u}_1,\theta_1) \quad (48)$$

Resorting to the dynamic program approach to this two-stage optimization process, the cost function is first minimized with respect to $\Delta\mathbf{u}_1$ and θ_1 (final stage control) using the single-stage optimization program described previously.

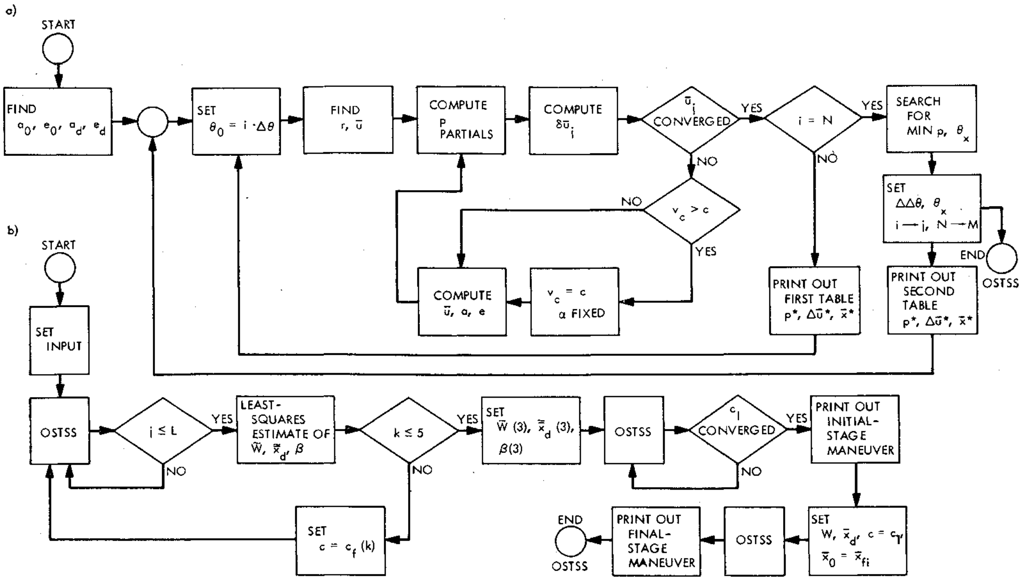

Fig. 4 Optimal stochastic transfer programs a) OS-TSS program, b) OSTTS program.

The basic assumption adopted in this paper is that the performance index thus minimized can be approximately represented (at least locally) in a quadratic form. Namely

$$p_{1i}(\mathbf{x}_1) \cong (\mathbf{x}_1 - \mathbf{x}_{1i})^T \bar{W}_{1i}(\mathbf{x}_1 - \tilde{x}_{1i}) + \beta_{1i} \qquad (49)$$

where β_{1i} is the minimum value at the ith local minimum, and it is approximately equal to f_0 in Eq. (39) evaluated at $\tilde{x}_{1i}$. When this is substituted into Eq. (48), the entire optimization process is reduced to

$$p_2(\mathbf{x}_0) = \min_{\theta_0, \Delta\mathbf{u}_0, i} [(\mathbf{x}_1 - \tilde{x}_{1i})^T \bar{W}_{1i}(\mathbf{x}_1 - \tilde{x}_{1i}) + \beta_{1i}] \qquad (50)$$

The new weighting matrix $\bar{W}_{1i}$, target vector $\tilde{x}_{1i}$ and constant term β_{1i} are determined by the least square method after having run the single-stage program at the sufficient number of points on the $\mathbf{x}_1$ vector space (the intermediate orbit).

This two-stage program is called the "Optimal Stochastic Transfer—Two Stage (OSTTS) Program," and its flow chart is shown in Figure 4b.

In carrying out the optimization process of Eq. (50) with respect to i, it is useful to predetermine the reachable region from the initial orbit $\mathbf{x}_0$ by single correction. The boundary surface of such reachable region can be derived from the condition of tangency of the orbit $\mathbf{x}_1$ with $\mathbf{x}_0$.

The orbit $\mathbf{x}_1$ is obviously reachable from the orbit $\mathbf{x}_0$ by single impulse, when they have intersections. Suppose two orbits have no intersection, and one parameter, e.g., ω_1, of $\mathbf{x}_1$ is gradually changed until $\mathbf{x}_1$ gets intersection with $\mathbf{x}_0$. Then the first contact between $\mathbf{x}_0$ and $\mathbf{x}_1$ takes place when both become tangent to each other. Therefore, the condition of tangency between two orbits describes the boundaries of the reachable region, because they have intersections within the boundaries, thus reachable.

This condition of tangency can be derived by setting the equation of elliptical orbit for orbit $\mathbf{x}_0$ and $\mathbf{x}_1$,

$$r = a(1 - e^2)/[1 + e \cos(\theta - \omega)] \qquad (51)$$

respectively, and by equating them, because r is equal at the point of intersection. This normally provides two solutions for θ, since intersections take place at two points, if there are any.

However, under certain conditions there is only one solution for θ and this is the condition for tangency, described in the following equation.

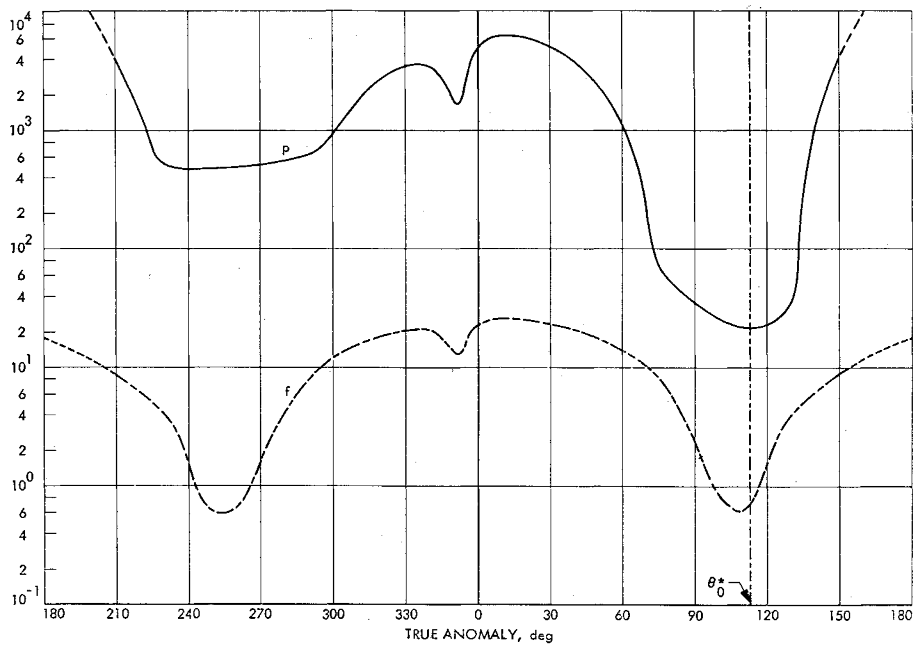

Fig. 5 Minimized performance index p and stochastic dispersion factor $f(\omega_0 = 5°)$.

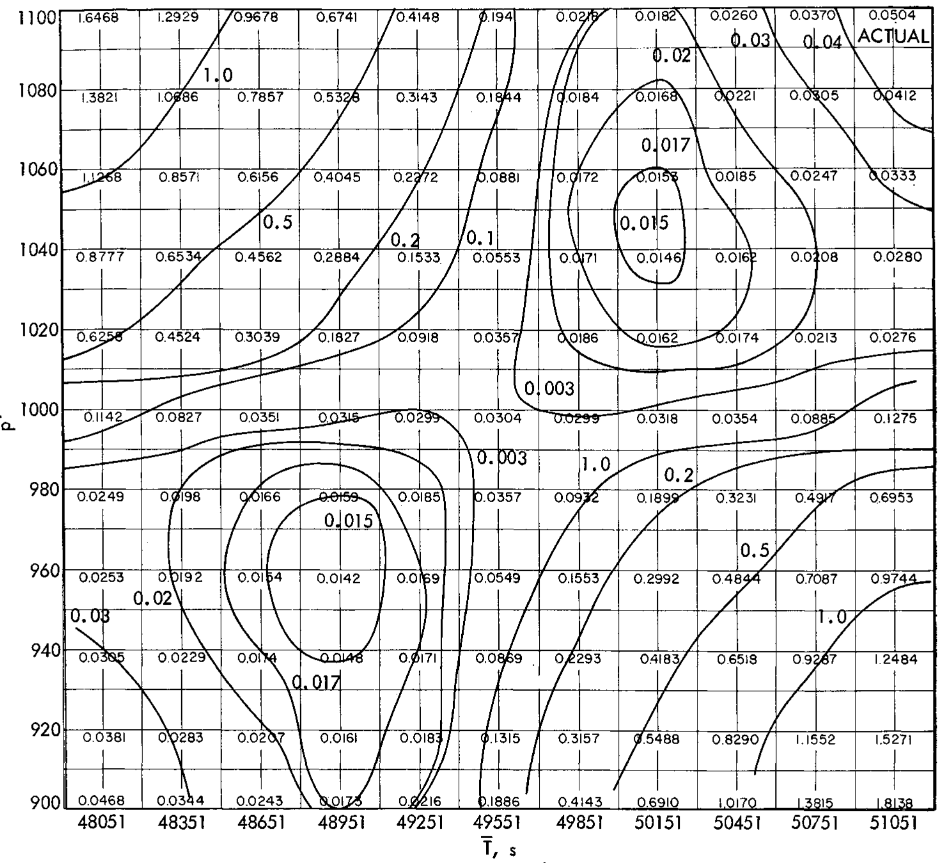

Fig. 6 Performance index contour map (ω = 1°).

The detail of derivation is given in Ref. 1.

$$a_1 = \{(h_{p1} + r_p)^2 + 2(a_0 - h_{p0} - r_p)(h_{p1} + r_p)\cos(\omega_1 - \omega_0) - [2a_0 - (h_{p0} + r_p)](h_{p0} + r_p)\}/2[h_{p1} + r_p + (a_0 - h_{p0} - r_p)\cos(\omega_1 - \omega_0) - a_0] : a_1 > 0, h_{p1} > 0 \quad (52)$$

and a_1 is related to T_1 by Eq. (13).

Those local minima which lie inside the boundary surface (with respect to $\mathbf{x}_0$) should be taken care of in the optimization of Eq. (50), while those which are away from the surface in the exterior region can be excluded because they are not reachable from $\mathbf{x}_0$ by single impulse.

Example

A Mars orbiter type mission was picked as an example for the two-stage orbit transfer program. The target orbit was T_d = 49551 sec, h_{pd} = 1.000 km and ω_d = 0°, while the initial orbit was 4346 sec in T and 100 km in h larger than the target, having its axis aligned.

The choice of the weighting matrix was done in such a way that a very heavy weight was imposed on the period T while a medium weight was placed on the periapsis altitude h_p. The weight on the periapsis argument ω was made very small. Then the deviations ΔT, Δh_p, and $\Delta\omega$ that respectively give a unit contribution to the performance index become 31.6 sec, 10 km, and 0.1 rad, respectively.

For the sake of simplicity the estimation error was ignored, and the assigned propellant budget was chosen as c = 150 m/sec, so that practically no constraint was imposed on the correction capability at every stage. As for execution errors, a relatively large pointing error (1.5%) was used, while the scale factor error was kept at 0.3%.

Figure 5 is an example of the minimized performance index p and the stochastic dispersion factor f along the orbit for a single stage maneuver. We observe three local maxima and three local minima, in all six entremum points for p. This is the reason why the heuristic search method by making two tables was employed, as described in the preceding part, in order to exclude the possibility of reaching a local extremum point by searching over the entire range of θ, instead of using a more analytical approach such as the gradient technique.

Figure 6 is one cross section of the performance index surface where ω_1 is kept at 1°. The contours indicate the corresponding performance index on the $T_1 - h_{p1}$ plane (ω_1 = 1°). When we divide the plane by the vertical and horizontal center lines whose intersection corresponds to the target orbit $\mathbf{x}_d$, the second quadrant becomes the region where $T_1 < T_d$ but $h_{p1} > h_d$ while the fourth quadrant becomes the region where $T_1 > T_d$ but $h_{p1} < h_d$. These two quadrants are essentially the areas where two orbits $\mathbf{x}_1$ and $\mathbf{x}_d$ have intersections. Therefore the optimal maneuver normally takes place at one of the intersections and the direction of the correction vector is closer to the normal direction to the uncorrected velocity vector than to the tangential direction. This drastically increases the contribution from the pointing error to the deviations, especially in period, so that large values of the performance index (PI) resulted in these areas. On the other hand, very small values of PI are seen in the first and third quadrants where $T_1 > T_d$, $h_{p1} > h_{pd}$ and $T_1 < T_d$, $h_{p1} < h_{pd}$, respectively, so that the two orbits tend to have no intersections. When the orbit $\mathbf{x}_{1i}$ lies in these regions, a correction which is almost tangential to the $\mathbf{u}_1$ vector will carry the $\mathbf{x}_1$ orbit close to the target $\mathbf{x}_d$ probably with a slight rotation of the line of apsides. Since the first-order contribution to the orbital period of the pointing error will be nullified by the tangential correction, the major error source will vanish under the policy of putting a heavy weight on the period as in this example.

Also, local minima were observed in the first quadrant as well as in the third quadrant. However, the latter was discarded from consideration because it was outside the boundary surface computed from Eq. (52) and was unreachable from the initial orbit $\mathbf{x}_0$ by means of a single correction because $\mathbf{x}_0$ was located at a remote distance from the center ($\mathbf{x}_d$) in the first

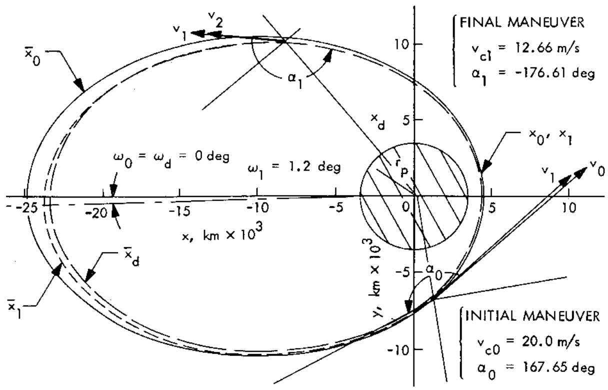

Fig. 7 Example of two-stage optimal maneuvers.

quadrant ($T_0 \gg T_d$, $h_{p0} \gg h_{pd}$, $\omega_0 = 0$). Therefore, the former local minimum point was picked up as the intermediate target orbit $\tilde{x}_1$ and the OSTTS program was applied. The $\tilde{x}_1$ was: $\tilde{T}_1 = 50{,}700$ sec, $\tilde{h}_{p1} = 1{,}062$ km, $\tilde{\omega}_1 = 1.20°$. The initial correction was $v_{c0} = 20.1$ m/sec, $\alpha_0 = 167.7°$, $\theta_0 = 279.5°$ and the resulted intermediate orbit $\mathbf{x}_1$ was: $T_1 = 50{,}701$ sec, $h_{p1} = 1062$ km, $\omega_1 = 1.20°$.

Then the final maneuver was executed at $\theta_1 = 129.0°$ with the magnitude $v_{c1} = 9.7$ m/sec and the direction $\alpha_1 = -174.62°$. As a result the final orbit $\mathbf{x}_2$ accomplishes perfect agreement with the target one in both period and periapsis altitude. But a slight off-set (0.3°) of orientation was observed.

In spite of this, the cost $p_2(\mathbf{x}_0)$ of these two maneuvers was only 0.026 because a very small weight was placed on ω. The results are depicted in Fig. 7.

As indicated in the figure, the optimal intermediate orbit $\mathbf{x}_1$ is an ellipse which is almost cotangential to both $\mathbf{x}_0$ and $\mathbf{x}_d$ (but shallowly intersecting with them) and whose line of apsides is slightly inclined ($\omega_1 = 1.2°$ while $\omega_0 = \omega_d = 0$). It is interesting to compare this transfer with the Hohmann-type transfer from the apogee of $\mathbf{x}_0$ to the perigee of $\mathbf{x}_d$. The latter is known as the transfer consuming the smallest amount of propellant[1] ($v_{c0} = 6.9$ m/sec, $v_{c1} = 19.5$ m/sec, total = 26.4 m/sec.). The over-all *PI* of the former is 0.026 whereas the *PI* of the final maneuver at the perigee of $\mathbf{x}_d$ is already 0.171. This is because of the scaling factor errors resulted from the large correction at the final stage (19.5 m/sec), even though the pointing error contribution is minimized by means of the tangential maneuver. Since the over-all *PI* of the latter transfer should exceed 0.171, it may be determined as inferior to the transfer of this example.

The standard deviations of orbital parameters are $\sigma_T = 15.06$ sec, $\sigma_{h_p} = 0.56$ km, $\sigma_\omega = 0.009$ deg. They describe the one-sigma range of uncertainty because of the execution errors of the initial maneuver. This is very small compared to the range where the curve-fitting technique was applied to evaluate the new weighting matrix. Hence, the stochastic averaging technique using this $\tilde{W}$ can be justified.

Conclusions

In this paper the dynamic programing algorithm was applied to the stochastic orbit transfer problem. The optimal imbedding of the dynamic programing technique was realized through a representation of the final stage performance index in a quadratic function. The major advantage of this assumption lies in the fact that the single-stage optimization program can be used for the initial stage maneuver because of the quadratic form of the new cost function, which is a large saving of computer time as well as memory space compared to the table formulating method.

Because of the highly nonlinear nature of the orbital transition matrix, the quadratic approximation of the performance index does not hold over a wide range of the $\mathbf{x}_1$ vector space. Therefore, the performance index is plotted as Fig. 6 over the reasonable range of parameter space of $\mathbf{x}_1$ and the local minima are detected there (two local minima in Fig. 6). Then excluding those which lie in the unreachable region, the coordinate $\tilde{x}_1$ is located at the center of the local minimum in the figure, and the new weighting matrix $\tilde{W}$ is computed applying the curve fitting technique to the local surface. This process is repeated for every local minimum and the optimization process of Eq. (50) is carried out for every i. Finally the over-all optimal two-stage strategy is determined by finding the minimum of $p_2(\mathbf{x}_0)$ with respect to i.

An extensive computer program implementing the previously described approach has been developed for use on the IBM 7094. Although this computer program is quite flexible and is capable of handling various combinations of mission objectives, the experimental results are reported specifically on the maneuver achieving the desired orbital period with high-precision subject to a relatively large pointing error. This is essential to the picture taking mission of the planet surface from the orbiter. Regions of low penalty as well as high penalty are indicated on the $T - h_p$ plane using ω as a parameter. Then an optimal two maneuver strategy which is very close to a tangential one is demonstrated against the classical Hohmann-type transfer which only applies to deterministic cases.

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